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# Periodic wave solutions of the Boussinesq equation

**Yi Zhang, Ling-ya Ye, Yi-neng Lv and Hai-qiong Zhao**

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, People's Republic of China

E-mail: zy2836@163.com

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**Abstract**

The Boussinesq equation usually arises in a physical problem as a long wave equation. The present work extends the search of periodic wave solutions for it. The Hirota bilinear method and Riemann theta function are employed in the process. We also analyse the asymptotic property of periodic waves in detail. Furthermore, it is of interest to note that well-known soliton solutions can be reduced from the periodic wave solutions.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

It is well known that the construction of explicit solutions for soliton equations is an important task in soliton theory. Especially, it is an important tool in characterizing many complicated phenomena and dynamical processes in physics, mechanics, chemistry, biology, etc. Some interesting explicit solutions have been found over recent decades, the most important among which are $N$-soliton solutions, quasi-periodic (or algebro-geometric) solutions, rational solutions, polar expansion solutions and others. In the process of searching for the solutions, quite a few systematic methods have been developed, such as inverse scattering transformation [1, 2], Darboux transformation [3, 4], Hirota bilinear method [5–10], algebro-geometric method [11–15] and so on.

Among them, the bilinear method introduced by Hirota provides us with a comprehensive approach to construct exact solutions of nonlinear evolution equations (NEEs). The idea was to make a transformation into new variables, so that in these new variables multisoliton solutions appear in a particularly simple form. The Hirota method turned out to be very effective and was quickly shown to give $N$-soliton solutions to the many important NEEs. The appeal and success of this approach lies in the fact that it allows one to obtain multisoliton solutions in a straightforward way. In recent years, the method also has been developed for obtaining Wronskian and Pfaffian forms of $N$-soliton solutions [10].

In the studies of the soliton theory, the algebro-geometric method, which was first developed by Matveev, Its, Novikov and Dubrovin *et al*, is also a powerful tool to construct the exact solutions of soliton equations. The exact solutions derived by this method are called quasi-periodic or algebro-geometric solutions, which can be used to find multisoliton solutions through the degeneracy procedure [16]. Very recently, algebro-geometric method has been successfully applied to obtain quasi-periodic solutions of several soliton equations [17–23]. It is noted that soliton solutions are typically expressed in terms of rational or hyperbolic functions, whereas quasi-periodic solutions require the use of Riemann theta functions and calculus on Riemann surfaces.

In this paper, we will use the Boussinesq equation [24]

$$u_{tt} - u_{xx} - (3u^2)_{xx} - u_{xxxx} = 0 \tag{1.1}$$

as a model to illustrate this idea. It is known that the propagation of long waves in shallow water is governed by the Boussinesq equation, which also arises in several other physical applications including one-dimensional nonlinear lattice waves, vibrations in a nonlinear string and ion sound waves in a plasma. In the particular field of water waves, the Boussinesq equation describes waves that are moving in one dimension but which may propagate in opposite directions. The Hirota bilinear operator and Riemann theta function will be employed to construct periodic wave solutions directly in the present paper. They have been demonstrated to be effective in treating solitary and periodic waves in the field of nonlinear waves. Indeed, the Hirota method enables us to find explicit periodic wave solutions by using complicated algebro-geometric theory. It is shown that the periodic wave solutions can be reduced to classical soliton solutions under a certain limit and the known results of solitary waves are recovered. Moreover, all parameters appearing in the solutions are free variables, whereas usual quasi-periodic solutions involve Riemann constants which are difficult to be determined and need to make complicated Abel transformation on the Riemann surface.

The objective of this paper is to study exact and explicit periodic wave solutions of the Boussinesq equation. As we all know, Nakamura, Fan and his co-workers have obtained periodic wave solutions of the KdV and KP equations by the bilinear approach [26, 27]. This gives us a way to present a direct method of calculating new exact solutions in Hirota's formalism. It is well known that not much work has been done on the solutions of the Boussinesq equation, except for soliton and fewer exact solutions [3, 24, 25]. Therefore, the study of periodic wave solutions will certainly enrich the theory of the Boussinesq equation. The outline of the present paper is as follows. In section 2, the Boussinesq equation is considered and the periodic wave solutions of this equation are presented. In particular, we obtain the one-periodic wave solution and two-periodic wave solutions. It is worthwhile to note that they can be reduced to classical one-soliton solution and two-soliton solutions, respectively. Finally, a summary and discussions are given in section 3.

## 2. Analysis

We first recall briefly the Boussinesq equation (1.1) in the light of our bilinear procedure. Through the dependent variable transformation

$$u = 2(\ln f)_{xx}, \tag{2.1}$$

equation (1.1) can be transformed into the following bilinear form:

$$\left(D_t^2 - D_x^2 - D_x^4\right) f \cdot f = 0, \tag{2.2}$$

where, as usual, $D_x$ and $D_t$ are the bilinear operators defined by

$$D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')|_{x'=x,t'=t}.$$

The first two-soliton solutions we obtain are given by

$$f_1 = 1 + e^{\xi_1} \quad \text{(solitary wave)}; \tag{2.3a}$$

$$f_2 = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1+\xi_2+A_{12}} \quad \text{(two-soliton)}, \tag{2.3b}$$

where

$$\xi_j = k_j x + \epsilon_j k_j \sqrt{1 + k_j^2}\, t + \xi_j^{(0)}, \qquad \epsilon_j = \pm 1, \quad j = 1, 2 \tag{2.3c}$$

and

$$e^{A_{12}} = \frac{3(k_1 - k_2)^2 + \left(\epsilon_1\sqrt{1 + k_1^2} - \epsilon_2\sqrt{1 + k_2^2}\right)^2}{3(k_1 + k_2)^2 + \left(\epsilon_1\sqrt{1 + k_1^2} - \epsilon_2\sqrt{1 + k_2^2}\right)^2}. \tag{2.3d}$$

In addition, let us consider equation (1.1) with the nonzero asymptotic condition, $u \longrightarrow u_0$ as $|x| \longrightarrow \infty$. Hence we look for a solution $u$ of the form

$$u = u_0 + 2(\ln f)_{xx} \tag{2.4}$$

by adding the constant solution $u_0$. Substituting (2.4) into equation (1.1) and integrating once again, we then get another bilinear form

$$G(D_x, D_t) f \cdot f = \left[D_t^2 - (1 + 6u_0)D_x^2 - D_x^4 + c\right] f \cdot f = 0, \tag{2.5}$$

where $c = c(t)$ is an integration constant.

It is important to note that the $D$-operator has good property when acting on exponential functions

$$D_x^m D_t^n\, e^{\xi_1} \cdot e^{\xi_2} = (k_1 - k_2)^m (\omega_1 - \omega_2)^n\, e^{\xi_1+\xi_2},$$

where $\xi_j = k_j x + \omega_j t + \xi_j^{(0)}$, $j = 1, 2$. More general, we have

$$G(D_x, D_t)\, e^{\xi_1} \cdot e^{\xi_2} = G(k_1 - k_2, \omega_1 - \omega_2)\, e^{\xi_1+\xi_2}.$$

Afterwards, we consider the Riemann theta function solution of the Boussinesq equation

$$f = \sum_{n \in Z^N} e^{\pi i \langle \tau n, n\rangle + 2\pi i \langle \xi, n\rangle}, \tag{2.6}$$

where $n = (n_1, \ldots, n_N)^{\mathrm{T}}, \xi = (\xi_1, \ldots, \xi_N)$, $\tau$ is a symmetric matrix and $\mathrm{Im}|\tau| > 0$, $\xi_j = k_j x + \omega_j t + \xi_j^{(0)}$, $j = 1, \ldots, N$.

In what follows, we present a one-periodic wave solution and two-periodic wave solutions of the Boussinesq equation in two cases of $N = 1$ and $N = 2$.

### *2.1. One-periodic wave solution and its reduction*

Our concern here is with the case $N = 1$; then (2.6) becomes

$$f = \sum_{n=-\infty}^{\infty} e^{2\pi i n\xi + \pi i n^2 \tau}. \tag{2.7}$$

Substituting (2.7) into (2.5) gives

$$\begin{aligned} Gf \cdot f &= G(D_x, D_t) \sum_{n=-\infty}^{\infty} e^{2\pi i n\xi + \pi i n^2\tau} \cdot \sum_{m=-\infty}^{\infty} e^{2\pi i m\xi + \pi i m^2\tau} \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G(D_x, D_t)\, e^{2\pi i n\xi + \pi i n^2\tau} \cdot e^{2\pi i m\xi + \pi i m^2\tau} \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G[2\pi i(n-m)k, 2\pi i(n-m)\omega]\, e^{2\pi i(n+m)\xi + \pi i(n^2+m^2)\tau}$$

$$\overset{n+m=m'}{=} \sum_{m'=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} G[2\pi i(2n-m')k, 2\pi i(2n-m')\omega]\, e^{\pi i[n^2+(n-m')^2]\tau} \right\} e^{2\pi i m'\xi}$$

$$\equiv \sum_{m'=-\infty}^{\infty} \overline{G}(m')\, e^{2\pi i m'\xi} = 0.$$

Note that

$$\begin{aligned}
\overline{G}(m') &= \sum_{n=-\infty}^{\infty} G[2\pi i(2n-m')k, 2\pi i(2n-m')\omega]\, e^{\pi i[n^2+(n-m')^2]\tau} \\
&\overset{n=n'+1}{=} \sum_{n'=-\infty}^{\infty} G\{2\pi i[2n'-(m'-2)]k, 2\pi i[2n'-(m'-2)]\omega\} \\
&\quad \times \exp\{\pi i[n'^2 + (n'-(m'-2))^2]\tau\} \exp[2\pi i(m'-1)\tau] \\
&= \overline{G}(m'-2)\, e^{2\pi i(m'-1)\tau} \\
&= \cdots = \begin{cases} \overline{G}(0)\, e^{\pi i m'^2\tau/2}, & m' \text{ is even} \\ \overline{G}(1)\, e^{\pi i(m'^2-1)\tau/4}, & m' \text{ is odd,} \end{cases}
\end{aligned}$$

which implies that if $\overline{G}(0) = \overline{G}(1) = 0$, then

$$\overline{G}(m') = 0, m' \in Z.$$

It is to note that (2.4) is an important class of exact solutions for the Boussinesq equation.

In this way, we may let

$$\overline{G}(0) = \sum_{n=-\infty}^{\infty} [-16\pi^2 n^2\omega^2 + 16(1+6u_0)\pi^2 n^2 k^2 - 256\pi^4 n^4 k^4 + c]\, e^{2\pi i n^2\tau} = 0, \tag{2.8}$$

$$\begin{aligned}
\overline{G}(1) = \sum_{n=-\infty}^{\infty} & [-4\pi^2(2n-1)^2\omega^2 + 4(1+6u_0)\pi^2(2n-1)^2 k^2 \\
& - 16\pi^4(2n-1)^4 k^4 + c]\, e^{\pi i(2n^2-2n+1)\tau} = 0.
\end{aligned} \tag{2.9}$$

Denote

$$\delta_1(n) = e^{2\pi i n^2\tau}, \qquad \delta_2(n) = e^{\pi i(2n^2-2n+1)\tau},$$

$$a_{11} = -\sum_{n=-\infty}^{\infty} 16\pi^2 n^2 \delta_1(n), \qquad a_{12} = \sum_{n=-\infty}^{\infty} \delta_1(n),$$

$$b_1 = \sum_{n=-\infty}^{\infty} [16(1+6u_0)\pi^2 n^2 k^2 - 256\pi^4 n^4 k^4]\delta_1(n), \quad a_{21} = -\sum_{n=-\infty}^{\infty} 4\pi^2(2n-1)^2\delta_2(n),$$

$$a_{22} = \sum_{n=-\infty}^{\infty} \delta_2(n), \quad b_2 = \sum_{n=-\infty}^{\infty} [4(1+6u_0)\pi^2(2n-1)^2 k^2 - 16\pi^4(2n-1)^4 k^4]\delta_2(n).$$

Then (2.8) and (2.9) can be written as

$$a_{11}\omega^2 + a_{12}c + b_1 = 0, \qquad a_{21}\omega^2 + a_{22}c + b_2 = 0.$$

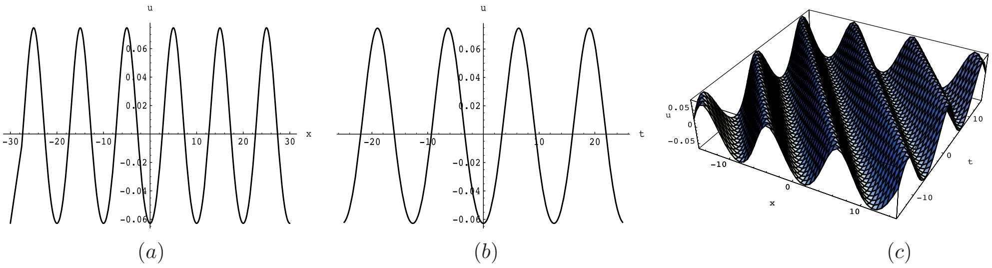



**Figure 1.** The plot of a one-periodic wave for the Boussinesq equation: (*a*) along the $x$-axis, (*b*) along the $t$-axis, (*c*) $u$ versus $x$ and $t$, where $k = 0.1$, $\tau = \mathrm{i}$.

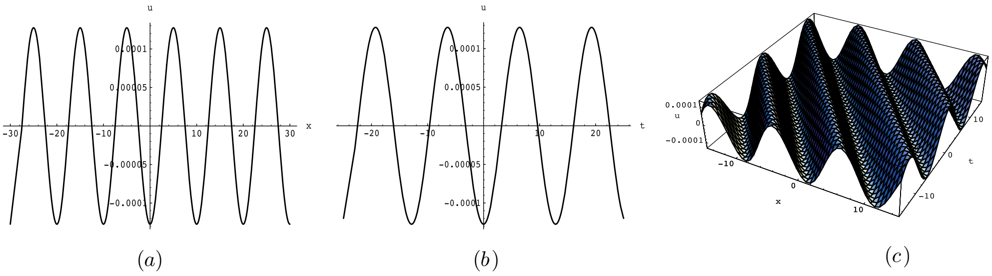



**Figure 2.** The plot of a one-periodic wave for the Boussinesq equation: (*a*) along the $x$-axis, (*b*) along the $t$-axis, (*c*) $u$ versus $x$ and $t$, where $k = 0.1$, $\tau = 3\mathrm{i}$.

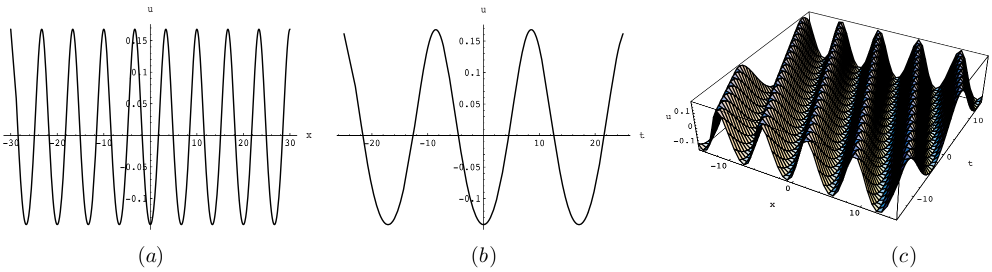



**Figure 3.** The plot of a one-periodic wave for the Boussinesq equation: (*a*) along the $x$-axis, (*b*) along the $t$-axis, (*c*) $u$ versus $x$ and $t$, where $k = 0.15$, $\tau = \mathrm{i}$.

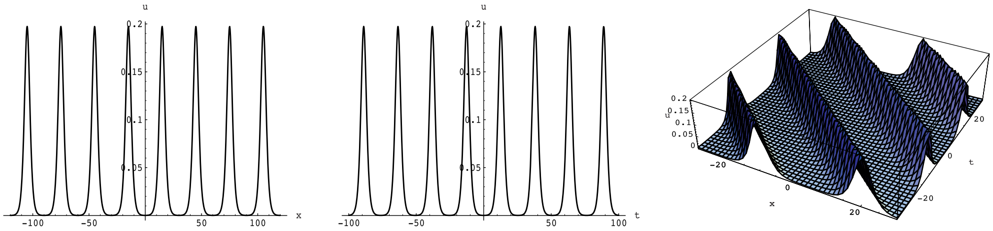
Solving this system, we get

$$\omega^2 = \frac{b_2 a_{12} - b_1 a_{22}}{a_{11} a_{22} - a_{12} a_{21}}, \qquad c = \frac{b_1 a_{21} - b_2 a_{11}}{a_{11} a_{22} - a_{12} a_{21}}. \tag{2.10}$$

Finally, we get a one-periodic wave solution

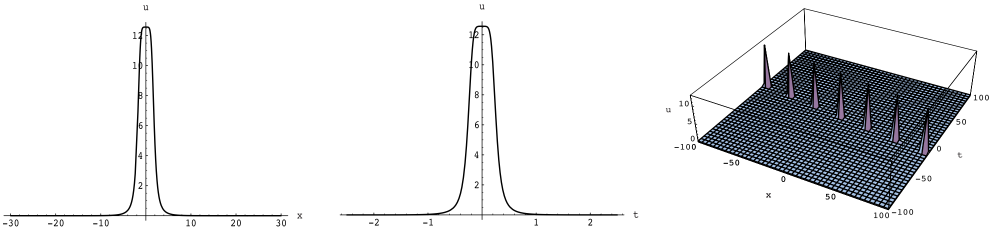
$$u = u_0 + 2(\ln f)_{xx}, \tag{2.11}$$

where $f$ and $\omega$ are given by (2.7) and (2.10), respectively. Figures 1–5 illustrate the five possibilities of a one-periodic wave for the Boussinesq equation by choosing the parameters $k$ and $\tau$ appropriately. It is important to emphasize that the solitons retain their identities, save for a cumulative phase shift. From figures 1–3, we can see that the parameter $\tau$ does not affect




(*a*) (*b*) (*c*)

**Figure 4.** The plot of a one-periodic wave for the Boussinesq equation: (*a*) along the *x*-axis, (*b*) along the *t*-axis, (*c*) $u$ versus $x$ and $t$, where $k = 0.1\mathrm{i}$, $\tau = 3\mathrm{i}$.




(*a*) (*b*) (*c*)

**Figure 5.** The plot of a one-periodic wave for the Boussinesq equation: (*a*) along the *x*-axis, (*b*) along the *t*-axis, (*c*) $u$ versus $x$ and $t$, where $k = 0.1\mathrm{i}$, $\tau = 0.01\mathrm{i}$.

the period and shape of the wave, while the parameter $k$ has an effect on the period and shape of the wave. But when $k$ is chosen as imaginary number, the parameter $\tau$ has influence on them, which are shown in figures 4 and 5. It would be of considerable interest to note that figure 5 pictures only one solitary wave, for a specific choice of the parameters $k = 0.1\mathrm{i}$ and $\tau = 0.01\mathrm{i}$. Another very important aspect to consider is for other choices of parameters; there will be in general other possibilities.

The well-known soliton solution of the Boussinesq equation can be obtained as a limit of the periodic solution (2.11). For this purpose, we write $f$ as

$$f = 1 + \alpha(\mathrm{e}^{2\pi\mathrm{i}\xi} + \mathrm{e}^{-2\pi\mathrm{i}\xi}) + \alpha^4(\mathrm{e}^{4\pi\mathrm{i}\xi} + \mathrm{e}^{-4\pi\mathrm{i}\xi}) + \cdots,$$

where $\alpha = \mathrm{e}^{\mathrm{i}\pi\tau}$.

Setting $u_0 = 0$, $\xi = \xi'/2\pi\mathrm{i} - \tau/2$, $k' = 2\pi\mathrm{i}k$, $\omega' = 2\pi\mathrm{i}\omega$, we get

$$\begin{aligned} f &= 1 + \mathrm{e}^{k'x+\omega't} + \alpha^2\,\mathrm{e}^{-\xi'} + \alpha^2\,\mathrm{e}^{2\xi'} + \alpha^6\,\mathrm{e}^{-2\xi'} + \cdots \\ &\longrightarrow 1 + \mathrm{e}^{k'x+\omega't}, \qquad \text{as} \quad \alpha \longrightarrow 0. \end{aligned}$$

So the periodic solution (2.11) can be reduced to the well-known soliton solution

$$u = 2(\ln f)_{xx}, \qquad f = 1 + \mathrm{e}^{k'x+\omega't},$$

which is shown to be equivalent to (2.3*a*). In order to satisfy (2.3*c*), we only need to prove that

$$\omega' \longrightarrow \epsilon k'\sqrt{1 + k'^2}, \qquad \epsilon = \pm 1, \qquad \text{as} \quad \alpha \longrightarrow 0. \tag{2.12}$$

In fact, it is not difficult to obtain that

$$\begin{aligned} a_{11} &= -32\pi^2(\alpha^2 + 4\alpha^8 + \cdots), \\ a_{12} &= 1 + 2\alpha^2 + \cdots, \end{aligned}$$

$$a_{21} = -8\pi^2(\alpha + 9\alpha^5 + \cdots),$$
$$a_{22} = 2(\alpha + \alpha^5 + \cdots),$$
$$b_1 = 32k^2\pi^2(1 - 16k^2\pi^2)\alpha^2 + 128k^2\pi^2(1 - 64k^2\pi^2)\alpha^8 + \cdots,$$
$$b_2 = 8k^2\pi^2(1 - 4k^2\pi^2)\alpha + 72k^2\pi^2(1 - 36k^2\pi^2)\alpha^5 + \cdots,$$

which lead to

$$b_2a_{12} - b_1a_{22} = 8k^2\pi^2(1 - 4k^2\pi^2)\alpha + o(\alpha^2), \qquad a_{11}a_{22} - a_{12}a_{21} = 8\pi^2\alpha + o(\alpha^2).$$

Therefore, we have

$$\omega \longrightarrow \epsilon k\sqrt{1 - 4k^2\pi^2}, \qquad \epsilon = \pm 1, \qquad \text{as} \quad \alpha \longrightarrow 0,$$

which implies (2.12).

### 2.2. Two-periodic wave solution and its reduction

In this section, let us consider two-periodic wave solutions of the Boussinesq equation ($N = 2$). Substituting (2.6) into (2.5), we have

$$\begin{aligned}
Gf \cdot f &= \sum_{n,m\in Z^2} G(D_x, D_t)\, \mathrm{e}^{2\pi \mathrm{i}\langle \xi,n\rangle + \pi \mathrm{i}\langle \tau n,n\rangle} \cdot \mathrm{e}^{2\pi \mathrm{i}\langle \xi,m\rangle + \pi \mathrm{i}\langle \tau m,m\rangle} \\
&= \sum_{n,m\in Z^2} G(2\pi \mathrm{i}\langle n - m, k\rangle, 2\pi \mathrm{i}\langle n - m, \omega\rangle)\, \mathrm{e}^{2\pi \mathrm{i}\langle \xi,n+m\rangle + \pi \mathrm{i}(\langle \tau m,m\rangle + \langle \tau n,n\rangle)} \\
&\overset{n+m=m'}{=} \sum_{m'\in Z^2} \sum_{n_1,n_2=-\infty}^{\infty} G(2\pi \mathrm{i}\langle 2n - m', k\rangle, 2\pi \mathrm{i}\langle 2n - m', \omega\rangle) \\
&\quad \times \exp[\pi \mathrm{i}(\langle \tau(n - m'), n - m'\rangle + \langle \tau n, n\rangle)] \exp(2\pi \mathrm{i}\langle \xi, m'\rangle) \\
&\equiv \sum_{m'\in Z^2} \overline{G}(m_1', m_2')\, \mathrm{e}^{2\pi \mathrm{i}\langle \xi,m'\rangle} = 0.
\end{aligned}$$

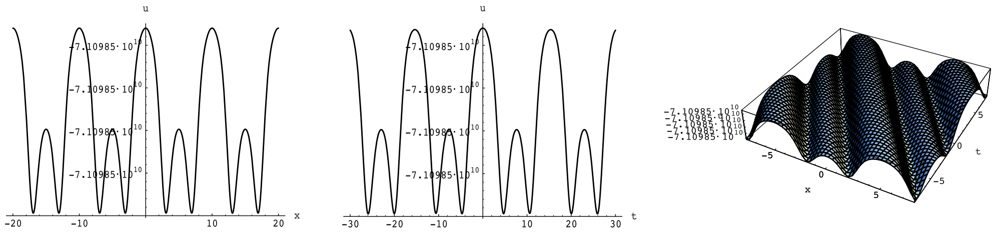
It is easy to calculate that

$$\begin{aligned}
\overline{G}(m_1', m_2') &= \sum_{n_1,n_2=-\infty}^{\infty} G(2\pi \mathrm{i}\langle 2n - m', k\rangle, 2\pi \mathrm{i}\langle 2n - m', \omega\rangle)\, \mathrm{e}^{\pi \mathrm{i}(\langle \tau(n-m'),n-m'\rangle + \langle \tau n,n\rangle)} \\
&\overset{n_j=n_j'+\delta_{jl},l=1,2}{=} \sum_{n_1,n_2=-\infty}^{\infty} G\left[2\pi \mathrm{i}\sum_{j=1}^{2}(2n_j' - (m_j' - 2\delta_{jl}))k_j,\, 2\pi \mathrm{i}\sum_{j=1}^{2}(2n_j' - (m_j' - 2\delta_{jl}))\omega_j\right] \\
&\quad \times \exp\left\{\pi \mathrm{i}\sum_{j,k=1}^{2}[(n_j' + \delta_{jl})\tau_{jk}(n_k' + \delta_{kl})\right. \\
&\quad \left. + ((m_j' - 2\delta_{jl} - n_j') + \delta_{jl})\tau_{jk}((m_k' - 2\delta_{kl} - n_k') + \delta_{kl})]\right\} \\
&= \begin{cases} \overline{G}(m_1' - 2, m_2')\, \mathrm{e}^{2\pi \mathrm{i}(m_1'-1)\tau_{11} + 2\pi \mathrm{i}m_2'\tau_{12}}, & l = 1 \\ \overline{G}(m_1', m_2' - 2)\, \mathrm{e}^{2\pi \mathrm{i}(m_2'-1)\tau_{22} + 2\pi \mathrm{i}m_1'\tau_{12}}, & l = 2, \end{cases}
\end{aligned}$$

which implies that if

$$\overline{G}(0, 0) = \overline{G}(0, 1) = \overline{G}(1, 0) = \overline{G}(1, 1) = 0,$$

then $\overline{G}(m_1', m_2') = 0$ and (2.4) is an exact solution of the Boussinesq equation.




(*a*) (*b*) (*c*)

**Figure 6.** The plot of a two-periodic wave for the Boussinesq equation: (*a*) along the *x*-axis, (*b*) along the *t*-axis, (*c*) $u$ versus $x$ and $t$, where $k_1 = 0.1, k_2 = -0.3, \tau_{11} = 0.1\mathrm{i}, \tau_{12} = 0.2\mathrm{i}, \tau_{22} = 3\mathrm{i}$.

Denote

$$
\begin{aligned}
a_{j1} &= -\sum_{n_1,n_2=-\infty}^{\infty} 4\pi^2\big(2n_1 - m_1^j\big)^2\delta_j(n),\\
a_{j2} &= -\sum_{n_1,n_2=-\infty}^{\infty} 4\pi^2\big(2n_2 - m_2^j\big)^2\delta_j(n),\\
a_{j3} &= \sum_{n_1,n_2=-\infty}^{\infty} 24\pi^2\langle 2n - m^j, k\rangle^2\delta_j(n),\\
a_{j4} &= \sum_{n_1,n_2=-\infty}^{\infty} \delta_j(n),\\
b_j &= \sum_{n_1,n_2=-\infty}^{\infty} (-4\pi^2\langle 2n - m^j, k\rangle^2 + 16\pi^4\langle 2n - m^j, k\rangle^4)\delta_j(n),\\
&\delta_j(n) = \mathrm{e}^{\pi\mathrm{i}\langle\tau(n-m^j),n-m^j\rangle+\pi\mathrm{i}\langle\tau n,n\rangle},\\
&j = 1, 2, 3, 4, m^1 = (0, 0), \quad m^2 = (1, 0), \quad m^3 = (0, 1), \quad m^4 = (1, 1)\\
&A = (a_{kj})_{4\times4}, b = (b_1, b_2, b_3, b_4)^{\mathrm{T}}.
\end{aligned}
$$

Then we have

$$
A\begin{pmatrix}\omega_1^2\\ \omega_2^2\\ u_0\\ c\end{pmatrix} = b,
$$

from which we obtain

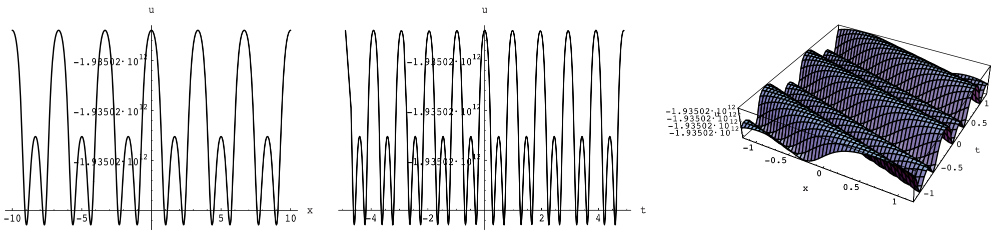
$$
\omega_1^2 = \frac{\Delta_1}{\Delta}, \qquad \omega_2^2 = \frac{\Delta_2}{\Delta}, \qquad u_0 = \frac{\Delta_3}{\Delta}, \tag{2.13}
$$

where $\Delta = |A|$ and $\Delta_1, \Delta_2, \Delta_3$ are produced from $\Delta$ by replacing its first, second and third column with $b$, respectively.

Finally, we get two-periodic wave solutions

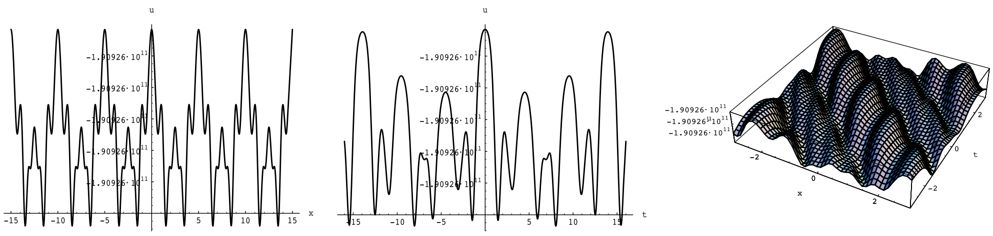
$$
u = u_0 + 2(\ln f)_{xx}, \tag{2.14}
$$

where $f$ and $\omega_1, \omega_2$ are given by (2.6) and (2.13), respectively. The properties of the solutions are shown in figures 6–10 for suitable parametric choices. Figures 6–9 give effect of the



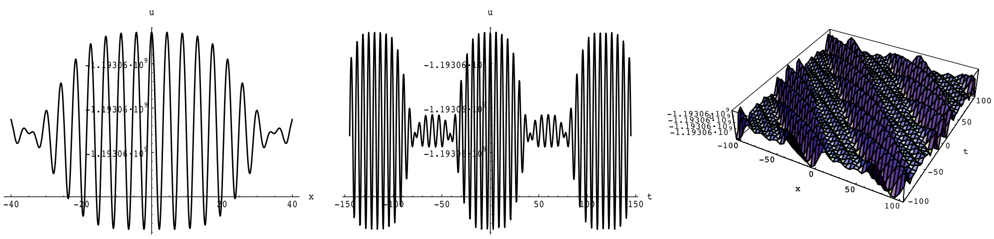

$(a)$ $(b)$ $(c)$

**Figure 7.** The plot of a two-periodic wave for the Boussinesq equation: ($a$) along the $x$-axis, ($b$) along the $t$-axis, ($c$) $u$ versus $x$ and $t$, where $k_1 = 0.3, k_2 = -0.3, \tau_{11} = 0.1\mathrm{i}, \tau_{12} = 0.2\mathrm{i}, \tau_{22} = 3\mathrm{i}$.




$(a)$ $(b)$ $(c)$

**Figure 8.** The plot of a two-periodic wave for the Boussinesq equation: ($a$) along the $x$-axis, ($b$) along the $t$-axis, ($c$) $u$ versus $x$ and $t$, where $k_1 = 0.2, k_2 = -0.2, \tau_{11} = 0.1\mathrm{i}, \tau_{12} = 0.3\mathrm{i}, \tau_{22} = 2\mathrm{i}$.




$(a)$ $(b)$ $(c)$

**Figure 9.** The plot of a two-periodic wave for the Boussinesq equation: ($a$) along the $x$-axis, ($b$) along the $t$-axis, ($c$) $u$ versus $x$ and $t$, where $k_1 = 0.01, k_2 = -0.2, \tau_{11} = 0.1\mathrm{i}, \tau_{12} = 0.3\mathrm{i}, \tau_{22} = 2\mathrm{i}$.

parameters $k_1, k_2$ and $\tau_{11}, \tau_{12}, \tau_{22}$ on the period and shape of waves. It is worthwhile to note that under suitable circumstances these parameters play a pivotal role in the shape and period of waves. It can also be straightforwardly seen that the amplitudes undergo changes dramatically for all other choices. Figure 10 shows that two-periodic waves can degenerate to a one-periodic wave when $k_1$ is sufficiently small.

The two-soliton solutions of the Boussinesq equation can be obtained as a limit of the periodic solutions (2.14). We write $f$ as

$$
\begin{aligned}
f = 1 &+ (\mathrm{e}^{2\pi\mathrm{i}\xi_1} + \mathrm{e}^{-2\pi\mathrm{i}\xi_1})\,\mathrm{e}^{\pi\mathrm{i}\tau_{11}} + (\mathrm{e}^{2\pi\mathrm{i}\xi_2} + \mathrm{e}^{-2\pi\mathrm{i}\xi_2})\,\mathrm{e}^{\pi\mathrm{i}\tau_{22}} \\
&+ (\mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2)} + \mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2)})\,\mathrm{e}^{\pi\mathrm{i}(\tau_{11}+2\tau_{12}+\tau_{22})} + \cdots.
\end{aligned}
$$

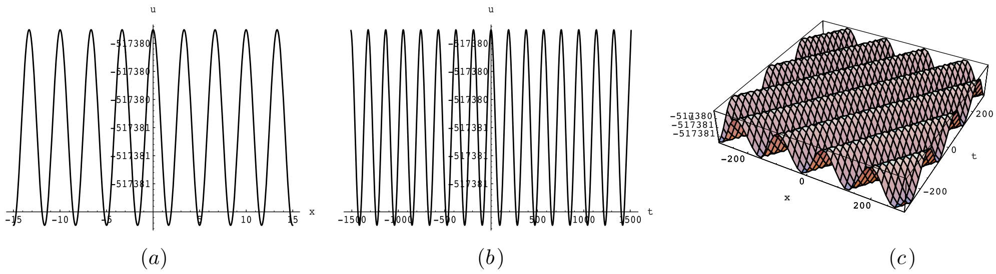


Figure 10. The plot of a two-periodic wave for the Boussinesq equation: ($a$) along the $x$-axis, ($b$) along the $t$-axis, ($c$) $u$ versus $x$ and $t$, where $k_1 = 0.0002$, $k_2 = -0.3$, $\tau_{11} = 0.1\mathrm{i}$, $\tau_{12} = 0.2\mathrm{i}$, $\tau_{22} = 3\mathrm{i}$.

Setting $\xi_1' = 2\pi \mathrm{i}\xi_1 + \pi\mathrm{i}\tau_{11}$, $\xi_2' = 2\pi\mathrm{i}\xi_2 + \pi\mathrm{i}\tau_{22}$, $\tau_{12} = i\widetilde{\tau}$ ($\widetilde{\tau}$ is a real), we get

$$\begin{aligned} f &= 1 + \mathrm{e}^{\xi_1'} + \mathrm{e}^{\xi_2'} + \mathrm{e}^{\xi_1'+\xi_2'+2\pi\mathrm{i}\tau_{12}} + \alpha_1^2\,\mathrm{e}^{-\xi_1'} + \alpha_2^2\,\mathrm{e}^{-\xi_2'} + \alpha_1^2\alpha_2^2\,\mathrm{e}^{-\xi_1'-\xi_2'+2\pi\mathrm{i}\tau_{12}} + \cdots \\ &\longrightarrow 1 + \mathrm{e}^{\xi_1'} + \mathrm{e}^{\xi_2'} + \mathrm{e}^{\xi_1'+\xi_2'-2\pi\widetilde{\tau}}, \text{ as } \alpha_1, \alpha_2 \longrightarrow 0, \end{aligned}$$

where

$$\alpha_1 = \mathrm{e}^{\pi\mathrm{i}\tau_{11}}, \qquad \alpha_2 = \mathrm{e}^{\pi\mathrm{i}\tau_{22}}, \qquad \xi_j' = k_j'x + \omega_j't + \pi\mathrm{i}\tau_{jj},$$

$$\mathrm{e}^{-2\pi\widetilde{\tau}} = \frac{3(k_1' - k_2')^2 + \left(\epsilon_1\sqrt{1+k_1'^2} - \epsilon_2\sqrt{1+k_2'^2}\right)^2}{3(k_1' + k_2')^2 + \left(\epsilon_1\sqrt{1+k_1'^2} - \epsilon_2\sqrt{1+k_2'^2}\right)^2},$$

$$\omega_j' \longrightarrow \epsilon_j k_j'\sqrt{1+k_j'^2}, \epsilon_j = \pm 1, \; j = 1, 2, \qquad \text{as} \quad \alpha_1, \alpha_2 \longrightarrow 0.$$

Obviously, it is seen to be equivalent to (2.3$b$)–(2.3$d$).

## 3. Conclusion and discussion

In this paper, based on the above study, exact and explicit periodic wave solutions of the Boussinesq equation have been presented by virtue of the Hirota bilinear method and the Riemann theta function. Moreover, they can be reduced to classical soliton solutions under a certain limit and known results of solitary waves are recovered. Regarding the conclusions of this equation, perhaps the most significant is that for choosing the constant solution $u_0$ appropriately. The result not only provides us an effective method to construct new exact solutions, but also greatly enriches the solution structure for the Boussinesq equation. It is an important aspect of the present work that the methods we have employed can be readily adapted to the other NEEs that possess the isospectral property and/or have soliton solutions obtainable by the direct approach. However, it remains an open problem as to whether the periodic wave solutions can be reduced to rational solutions, positon, negaton and complexiton solutions, and further investigation on it will be pursued in the future. We strongly believe that the bilinear approach will continue to be successful and more surprising results await our efforts.

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